SOLUTION OF A PLANE STEADY HEAT CONDUCTION PROBLEM WITH BOUNDARY CONDITIONS OF THE THIRD KIND FOR REGIONS OF SPECIAL TYPE

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It is shown that the steady problem of heat conduction theory for regions bounded by cochleas of order 4m + 2 (m = 1, 2, 3, ..., N), which emit heat from their surfaces according to Newton's law, is reduced by conformal mapping to the solution of certain equations in finite differences. For the case m = 1 the solution of the equations is expressed in terms of Bessel functions, and formulas for the temperature distribution are obtained.

INTRODUCTION

Certain steady problems in heat-conduction theory reduce to solution of the Laplace equation [9]

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ on D} \tag{1}$$

with the boundary condition

$$\frac{\partial u}{\partial n}+hu|_B=f(p), \tag{2}$$

where h is a positive constant, and $f(p)$ is a given function.

In general the boundary conditions (2) do not permit effective use of the method of conformal mapping for solving the problem. We may select a class of regions, however, for which the function accomplishes conformal mapping of the given region onto a circle, is represented by the Newtonian binomial of odd order [4]

$$W=R(\xi+\lambda)^{2m+1}+B, \tag{3}$$

where

$$m=1,\ 2,\ 3,\dots,N,\quad \xi=\rho\exp i\theta,\ \lambda\geqslant 1/\sin\frac{\pi}{2m+1}.$$

In this case the problem for the circle reduces to the solution of the equations in finite differences. Solution of these equations may be represented in the form of Laplace contour integrals. Curves corresponding to the unit circle in the ξ plane constitute a family of cochleas of high order.*

Using the equation of circle $\xi\bar{\xi}=1$, it is not difficult to obtain from (3) the equation of the curves in polar coordinates

$$r=R\left\{\lambda\cos\frac{\psi}{2m+1}\pm\sqrt{1-\lambda^2\sin^2\frac{\psi}{2m+1}}\right\}^{2m+1}, \tag{4}$$

where

$$\lambda\geqslant 1/\sin\frac{\pi}{2m+1},\quad m=1,\ 2,\ 3,\dots,N.$$

The length of the arc of curves (4) is a rational function of the parameter [9]:

$$S=\int_{-\pi}^{+\pi}\left|\frac{dW}{d\xi}\right|_{\rho=1}d\theta=(2m+1)R\int_{-\pi}^{+\pi}(1+2\lambda\cos\theta+\lambda^2)^m d\theta=$$
$$=2\pi R(2m+1)(\lambda^2-1)^m P_m\left(\frac{\lambda^2+1}{\lambda^2-1}\right), \tag{5}$$

where the P_m are Legendre polynomials.

1. FORMULATION OF THE PROBLEM FOR A CIRCLE AND ITS REDUCTION TO AN EQUATION IN FINITE DIFFERENCES

In conformal mapping of (3), Eq. (1) and boundary conditions (2) transform to the form [1]

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{\partial^2 u}{\partial\theta^2}=0 \text{ on the circle} \tag{6}$$

with the boundary condition

$$\frac{\partial u}{\partial\rho}+hR(2m+1)(1+2\lambda\cos\theta+\lambda^2)^m u\Big|_{\rho=1}=f_1(\theta), \tag{7}$$

where $f_1(\theta)$ is a given function satisfying the Dirichlet conditions in the interval $-\pi\leq\theta\leq+\pi$. Then [3]

$$f_1(\theta)=\frac{b_0}{2}+\sum_{n=1}^{\infty}b_n\cos n\theta+c_n\sin n\theta. \tag{8}$$

The special feature of the case being examined, which allows an exact solution to be obtained, consists of the fact that the coefficient

$$h(\theta)=h\left|\frac{dW}{d\xi}\right|_{\rho=1}=hR(2m+1)(1+2\lambda\cos\theta+\lambda^2)^m \tag{9}$$

*The order of the curves is equal to 4m + 2. The case m = 1/2 corresponds to the family of Pascal cochleas.

is a trigonometric polynomial:

$$h(\theta)=\sum_{k=0}^{m}a_k\cos k\theta>0,\quad -\pi\leqslant\theta\leqslant+\pi,\qquad(10)$$

where

$$a_k=\frac{2}{\pi}\int_0^{\pi}h(\theta)\cos k\theta\,d\theta,\quad a_0=\frac{1}{\pi}\int_0^{\pi}h(\theta)\,d\theta.\qquad(11)$$

To evaluate the coefficients (11) we shall use the equality [3]

$$\frac{2}{\pi}\int_0^{\pi}(1+2\lambda\cos\theta+\lambda^2)^m\cos k\theta\,d\theta=$$

$$=2(\lambda^2-1)^m\frac{\Gamma(m+1)}{\Gamma(m+k+1)}P_m^k\left(\frac{\lambda^2+1}{\lambda^2-1}\right).\qquad(12)$$

Using (5), (11), and (12), we obtain

$$h(\theta)=\frac{hS}{2\pi}\left\{1+\frac{2\,\Gamma(m+1)}{P_m[(\lambda^2+1)/(\lambda^2-1)]}\times\right.$$

$$\left.\times\sum_{k=1}^{m}\frac{P_m^k[(\lambda^2+1)/(\lambda^2-1)]}{\Gamma(m+k+1)}\cos k\theta.\right\},\qquad(13)$$

where S is the length of arc of the curve; m = 1, 2, 3, ..., N; λ is a parameter; h is the heat-transfer coefficient; and P_m^k are the associated Legendre functions. We shall seek a solution of the problem (6), (7) in the form

$$u=A_0/2+\sum_{n=1}^{\infty}\rho^n[A_n\cos n\theta+B_n\sin n\theta].\qquad(14)$$

Substituting (14) in boundary conditions (7), we obtain for determination of the expansion coefficients the system of difference equations:

$$(n+a_0)A_n+\frac{1}{2}\sum_{k=1}^{m}a_k[A_{n+k}+A_{n-k}]=b_n,$$

$$n=0,1,2,3,\ldots,\qquad(15)$$

$$(n+a_0)B_n+\frac{1}{2}\sum_{k=1}^{m}a_k[B_{n+k}+B_{n-k}]=c_n,$$

$$n=1,2,3,\ldots\qquad(16)$$

with the condition

$$A_n\to 0\quad B_n\to 0$$
$$n\to\infty\quad n\to\infty\qquad(17)$$

$$A_n=A_{-n}\quad n=1,2,3,\ldots,m,\qquad(18)$$

$$B_n=-B_{-n}\quad n=0,1,2,3,\ldots,m-1.\qquad(19)$$

Solution of equations in finite differences of type (15) and (16) has been examined in [5, 7]. Conditions (17)–(19) allow the expansion coefficients A_n and B_n to be determined uniquely.

2. SOLUTION OF EQUATIONS IN FINITE DIFFERENCES

It is known that the general solution of the difference equations (15) and (16) has the form:*

$$A_n=\overline{A}_n+\sum_{k=1}^{2m}\omega_k A_k(n),\qquad(20)$$

where $\overline{A}_n$ is a particular solution of the inhomogeneous equation; $A_k(n)$ are linearly independent solutions of the homogeneous equation; ω_k are arbitrary periodic functions with unit period.

Following the method of Laplace, we shall seek a solution of (15) in the form

$$A_n=\int_\gamma t^{-n-1}V(t)\,dt.\qquad(21)$$

Integrating by parts, we obtain

$$nA_n=\int_\gamma t^{-n-1}tV'(t)\,dt-t^{-n}V(t)\Big|_\gamma,\qquad(22)$$

$$A_{n+k}=\int_\gamma t^{-n-1}t^{-k}V(t)\,dt,\ k=\pm1,\mp2,\ldots,\mp m.\qquad(23)$$

Let**

$$b_n=\int_\gamma t^{-n-1}\varphi(t)\,dt.\qquad(24)$$

Substituting (21)–(24) into (15), we find

$$\int_\gamma t^{-n-1}\left\{tV'(t)+\frac{1}{2}\left[\sum_{k=0}^{m}a_k\left(t^k+\frac{1}{t^k}\right)\right]V(t)-\varphi(t)\right\}dt=$$

$$=t^{-n}V(t)\Big|_\gamma.\qquad(25)$$

We shall choose the contour of integration and the function V(t) from the conditions

$$t^{-n}V(t)\Big|_\gamma=0,\ tV'(t)+\frac{1}{2}\left[\sum_{k=0}^{m}a_k\left(t^k+\frac{1}{t^k}\right)\right]V(t)=\varphi(t).\qquad(26)$$

The solution of (26) will be

$$V(t)=F(t)+CV_0(t),\qquad(27)$$

$$V_0(t)=t^{-a_0}\exp\{-\Omega_m(t)\},\qquad(28)$$

where

$$F(t)=t^{-a_0}\exp\{-\Omega_m(t)\}\int^t\tau^{a_0-1}\varphi(\tau)\exp\{\Omega_m(\tau)\}d\tau,\qquad(29)$$

$$\Omega_m(t)=\frac{1}{2}\sum_{k=1}^{m}\frac{a_k}{k}\left(t^k-\frac{1}{t^k}\right),\ a_m=|a_m|\exp i\alpha.\qquad(30)$$

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*Equations (15) and (16) differ only in the "initial" conditions, and it is therefore sufficient to examine one of them.

**For this it is sufficient that $f_1(\theta)$ satisfies the Dirichlet conditions.

In the plane t we may choose 2m rays on which function (28) tends to zero for any a_k (k = 0, 1, 2, 3, ..., m) [5]:

$$\varphi_k = \frac{2\pi k}{m} - \frac{\alpha}{m} \quad |k = 0, 1, 2, \ldots, m-1|$$

$$V_0(\rho \exp i\varphi_k) \underset{\rho\to\infty}{\to} 0, \tag{31}$$

$$\psi_k = \frac{(2k+1)\pi}{m} - \frac{\alpha}{m} \quad |k = 0, 1, 2, \ldots, m-1|$$

$$V_0(\rho \exp i\psi_k) \underset{\rho\to 0}{\to} 0. \tag{32}$$

We shall designate by γ_{k+1} the integration paths located inside the sector $\varphi_k \le \varphi \le \varphi_{k+1}$ $(0 < \rho \le \infty)$, and by Γ_{k+1} paths located inside the sector $\psi_k \le \varphi \le \le \psi_{k+1}$ $(0 \le \rho < \infty)$. γ_0 consists of the unit circle and parts of the section $\varphi = 0$, $\varphi = 2\pi$ $|1 \le \rho \le \infty|$. Then the solution of the equations (15) will be*

$$A_k(n) = \frac{m}{2\pi i}\int_{\gamma_k} t^{-n-1} V_0(t)\,dt, \tag{33}$$

$$A^{(k)}(n) = \frac{m}{2\pi i}\int_{\Gamma_k} t^{-n-1} V_0(t)\,dt, \quad k = 1, 2, 3, \ldots, m, \tag{34}$$

$$\overline{A}_n = \frac{1}{2\pi i}\int_{\gamma_0} t^{-n-1} F(t)\,dt. \tag{35}$$

For the integrals (33)–(35) we may obtain an asymptotic representation for large n [8]:

$$A_k(n) \underset{n\to\infty}{\to} \exp\left\{\frac{2k-1}{m}\pi i(n+a_0)\right\} \times \left[\left(\frac{a_m}{2m}\right)^{\frac{n+a_0}{m}} \Big/ \Gamma\left(\frac{n+a_0}{m}+1\right)\right], \tag{36}$$

$$A^{(k)}(n) \underset{n\to\infty}{\to} \exp\left\{\frac{2k}{m}\pi i(n+a_0)\right\}\left[\sin\frac{n+a_0}{m}\Gamma \Big/ \pi\right] \times \left[\Gamma\left(\frac{n+a_0}{m}\right) \Big/ \left(\frac{a_m}{2m}\right)^{\frac{n+a_0}{m}}\right], \tag{37}$$

$$\overline{A}_n \underset{n\to\infty}{\to} \frac{b_n}{n+a_0}, \quad \overline{B}_n \underset{n\to\infty}{\to} \frac{c_n}{n+a_0}, \text{ is not an integer.} \tag{38}$$

It follows from (36)–(38) that the limited solutions of (15) and (16) contain only m arbitrary periodic functions each:

$$A_n = \overline{A}_n + \sum_{k=1}^{m} \omega_k A_k(n+a_0), \tag{39}$$

$$B_n = \overline{B}_n + \sum_{k=1}^{m} \tilde{\omega}_k A_k(n+a_0). \tag{40}$$

*Since $V_0(\overline{t}) = \overline{V_0(t)}$, integrals taken along contours symmetrical relative to the real axis will be complex conjugates. When $\lambda > 0$ $\alpha = 0$.

From conditions (18) and (19) for ω_k and $\tilde{\omega}_k$ we have the system of algebraic equations*

$$\overline{A}_s - \overline{A}_{-s} + \sum_{k=1}^{m} \omega_k [A_k(a_0+s) - A_k(a_0-s)] = 0$$

$$(s = 1, 2, 3, \ldots, m), \tag{41}$$

$$\overline{B}_s + \overline{B}_{-s} + \sum_{k=1}^{m} \tilde{\omega}_k [A_k(a_0+s) + A_k(a_0-s)] = 0$$

$$(s = 0, 1, 2, 3, \ldots, m-1). \tag{42}$$

Thus, the expansion coefficients in (14) are determined from (39) and (40), where $A_k(n)$, $\overline{A}_n$, and $\overline{B}_n$ are given by the integrals (33)–(35), while the arbitrary periodic functions are found from the system of algebraic equations (41), (42) of order m.

EXAMPLES OF SOLUTION OF CERTAIN PROBLEMS OF MATHEMATICAL PHYSICS

Example 1. We shall examine the solution of the problem (1) for regions bounded by a family of cochleas of order 6. From (4) and (5) we have

$$r = \frac{S}{6\pi(1+\lambda^2)}\left\{\lambda\cos\frac{\psi}{3} \pm \sqrt{1-\lambda^2\sin^2\frac{\psi}{3}}\right\}^3,$$

$$\left| -3\arcsin\frac{1}{\lambda} \le \psi \le +3\arcsin\frac{1}{\lambda}\right|, \tag{43}$$

where $\lambda \ge 2/\sqrt{3}$ is a parameter, and S is the length of arc of the curve.

Under conformal mapping, we obtain

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{\partial^2 u}{\partial\theta^2} = 0 \text{ on the circle,} \tag{44}$$

with the boundary condition

$$\frac{\partial u}{\partial\rho} + \frac{hS}{2\pi}\left(1 + \frac{2\lambda}{1+\lambda^2}\cos\theta\right)u\Big|_{\rho=1} = f_1(\theta), \tag{45}$$

where $f_1(\theta)$ is assigned and satisfies the Dirichlet conditions

$$f_1(\theta) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n\cos n\theta + c_n\sin n\theta \quad |-\pi \le \theta \le +\pi|; \tag{46}$$

h is the heat-transfer coefficient;

$$a_0 = \frac{hS}{2\pi}; \quad a_1 = \frac{hS}{\pi}\frac{\lambda}{1+\lambda^2}; \tag{47}$$

$$u = \frac{A_0}{2} + \sum_{n=1}^{\infty} \rho^n [A_n\cos n\theta + B_n\sin n\theta]. \tag{48}$$

*It may be shown that the determinants of the system (41), (42) are not equal to zero under the conditions $h(\theta) > 0$ $|-\pi \le \theta \le +\pi|$ (see (10)).

The coefficients A_n and B_n are determined from the equations

$$(n + a_0) A_n + \frac{a_1}{2} [A_{n+1} + A_{n-1}] = b_n, \quad (49)$$

$$n = 0, 1, 2, 3, \ldots, \quad A_1 = A_{-1}, \quad (50)$$

$$(n + a_0) B_n + \frac{a_1}{2} [B_{n+1} + B_{n-1}] = c_n, \quad (51)$$

$$n = 1, 2, 3, \ldots, \quad B_0 = 0. \quad (52)$$

The solution of a homogeneous equation will be

$$A(n) = (-1)^n [\omega_1 I_{n+a_0}(a_1) + \omega_2 Y_{n+a_0}(a_1)], \quad (53)$$

where $I_\nu(a_1)$, $Y_\nu(a_1)$ are cylindrical functions of order 1 and 2.

From the condition of convergence of the series (48) we should put $\omega_2 \equiv 0$ [3], and then we have

$$A_n = \overline{A}_n + \omega_1 (-1)^n I_{n+a_0}(a_1), \quad (54)$$

$$B_n = \overline{B}_n + \tilde{\omega}_1 (-1)^n I_{n+a_0}(a_1). \quad (55)$$

From conditions (50) and (52) we obtain

$$\omega_1 = (\overline{A}_{-1} - \overline{A}_1)/2I'_{a_0}(a_1), \quad (56)$$

$$\tilde{\omega}_1 = -\overline{B}_0/I_{a_0}(a_1). \quad (57)$$

The denominators of (56) and (57) are not equal to zero.*

The particular solutions $\overline{A}_n$ and $\overline{B}_n$ have been constructed in the Appendix (A2–A7), where it is shown $\overline{A}_n$ and $\overline{B}_n$ are represented through the Green's function of the equation in finite differences.

Example 2. To find the temperature distribution within the region bounded by curve (43), where there is uniform liberation of heat. At the boundary the heat is radiated according to Newton's law. The temperature of the external medium is equal to zero [2].

The problem reduces to integration of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{Q}{K} \text{ on } D \quad (58)$$

with the boundary condition

$$\left.\frac{\partial u}{\partial n} + hu\right|_B = 0, \quad (59)$$

where K is the thermal conductivity; Q is the quantity of heat liberated in unit volume; h is the heat-transfer coefficient.

*We recall that the least positive root of the functions $I_{a_0}(a_1)$ and $I'_{a_0}(a_1)$ is larger than a_0 [8]. Here $a_0 - a_1 = hS/2\pi\ (\lambda - 1)^2 > 0$. There are no roots.

We shall choose the special solution

$$u = u_1 + u_2, \text{ where } \Delta u_2 = -\frac{Q}{K} \text{ on } D, \left. u_2\right|_B = 0. \quad (60)$$

Then for u_1 we have

$$\Delta u_1 = 0 \text{ on } D, \left.\frac{\partial u_1}{\partial n} + hu_1\right|_B = -\left.\frac{\partial u_2}{\partial n}\right|_B. \quad (61)$$

We shall use the conformal representation

$$W = x + jy = \frac{S}{6\pi(1+\lambda^2)} (\xi^3 + 3\lambda\xi^2 + 3\lambda^2\xi), \quad (62)$$

where $\xi = \rho \exp i\theta$, $\lambda \geq 2/\sqrt{3}$, and S is the length of arc.

For solution of problem (60) we choose the particular solution

$$u_3 = -\frac{Q}{4K}(x^2 + y^2) = -\frac{Q}{4K} W \cdot \overline{W}. \quad (63)$$

Then for u_2 we obtain

$$u_2 = -\frac{QS^2}{144k\pi^2(1+\lambda^2)^2} \{(\rho^6 - 1) + 9\lambda^2(\rho^4 - 1) + 9\lambda^4(\rho^2 - 1) + [6\lambda(\rho^5 - \rho) + 18\lambda^3(\rho^3 - \rho)]\cos\theta + 6\lambda^2(\rho^4 - \rho^2)\cos 2\theta\}. \quad (64)$$

For the conformal representation (62) the equation and the boundary conditions (61) transform to the form

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u_1}{\partial\rho}\right) + \frac{\partial^2 u_1}{\partial\theta^2} = 0 \text{ on the circle}, \quad (65)$$

with the boundary condition

$$\left.\frac{\partial u_1}{\partial\rho} + (a_0 + a_1\cos\theta)u_1\right|_{\rho=1} = \frac{QS^2}{12\pi^2 K}\left[\frac{b_0}{2} + b_1\cos\theta + b_2\cos 2\theta\right], \quad (66)$$

where

$$a_0 = \frac{hS}{2\pi}; \quad a_1 = \frac{hS}{\pi}\frac{\lambda}{1+\lambda^2}; \quad (67)$$

$$b_0 = (1 + 6\lambda^2 + 3\lambda^4)/(1+\lambda^2)^2; \quad b_1 = (2\lambda + 3\lambda^3)/(1+\lambda^2)^2; \quad b_2 = \lambda^2/(1+\lambda^2)^2. \quad (68)$$

The solution of (65), (66) will be

$$u_1 = \frac{QS^2}{12\pi^2 K}\left\{\frac{b_1}{a_1} - \frac{1}{2}\left(\frac{2}{a_1}\right)^2 (a_0 + 1) b_2 + \frac{2b_2}{a_1}\rho\cos\theta + \omega_1\left[\frac{1}{2} I_{a_0}(a_1) + \sum_{n=1}^{\infty}(-1)^n I_{n+a_0}(a_1)\rho^n\cos n\theta\right]\right\}, \quad (69)$$

where

$$\omega_1 = [2b_0/a_1 - (2/a_1)^2 a_0 b_1 + (2/a_1)^3 a_0 (a_0+1) b_2 - 4b_2/a_1] \times \times [2I'_{a_0}(a_1)]^{-1}$$

(see (54), (55) and (A5)–(A7)).

CONCLUSIONS

In solving a plane problem of potential theory with boundary conditions of the third kind [3] we use the method of conformal mappings onto the unit circle. The corresponding problem for a circle reduces to solution of linear equations in finite differences. The class of regions has been examined in which the modulus of the derivative of the mapping function is a trigonometric polynomial. In this case the exact solution of the problem is represented in the form of Laplace contour integrals.

For a rather wide class of regions we can obtain an approximate solution by restricting ourselves to a finite number of terms in the Fourier series for the modulus of the derivative of the mapping function.

The method of difference equations may also be used in solving problems for a circle with mixed boundary conditions with variable coefficients.

APPENDIX

We shall show that the particular solution of the inhomogeneous difference equation (49), (51) is represented through the Green's function of this equation. To improve the convergence of the series (48), (69) it is convenient to use a transformation of the Green's function either in the form of an expansion in terms of Bessel functions with integral index, or in the form of a discontinuous function. The Green's function of (49) satisfies the relations

$$(n+a_0) g_{mn} + \frac{a_1}{2} [g_{mn+1} + g_{mn-1}] = \delta_{mn}, \qquad \text{(A1)}$$

where

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n. \end{cases}$$

Then the particular solution of (49) and (52) is written in the form

$$\bar{A}_n = \sum_{m=0}^{\infty} b_m g_{mn}, \quad \bar{B}_n = \sum_{m=1}^{\infty} c_m g_{mn}. \qquad \text{(A2)}$$

From (35) and (29) we may obtain*

$$g_{mn} = \frac{1}{2\pi i} \int_{\gamma_0} t^{-a_0-n-1} \exp\left\{-\frac{a_1}{2}\left(t - \frac{1}{t}\right)\right\} \times \times \int^{t} \tau^{a_0+m-1} \exp\left\{\frac{a_1}{2}\left(\tau - \frac{1}{\tau}\right)\right\} d\tau\, dt. \qquad \text{(A3)}$$

*In (29) it is evidently sufficient to put $\varphi(t) = t^m/2\pi i$.

The integral (A3) allows the expansion

$$g_{mn} = \sum_{s=-\infty}^{s=+\infty} \frac{I_{s-m}(a_1) I_{n-s}(-a_1)}{s + a_0}, \quad a_0 \text{ is not an integer};$$

$$g_{mn} = \sum_{s=-\infty}^{s=+\infty}{}' \frac{I_{s-m}(a_1) I_{n-s}(-a_1)}{s+j} - - I_{-m-j}(a_1) \left.\frac{\partial I_\nu(-a_1)}{\partial \nu}\right|_{\nu=n+j} \quad a_0 = j. \qquad \text{(A4)}$$

Using the Lagrange method to solve (A1), we obtain another representation of the Green's function

$$g_{mn} = \pi(-1)^{m+n} \{ Y_{n+a_0}(a_1) I_{m+a_0}(a_1) - Y_{m+a_0}(a_1) I_{n+a_0}(a_1) \},$$

$$n \leqslant m - 1;$$

$$g_{mn} = 0, \; n > m - 1, \qquad \text{(A5)}$$

whence

$$g_{mm} = 0, \; g_{mm-1} = 2/a_1, \; g_{mm+1} = 0. \qquad \text{(A6)}$$

<u>Example</u>. Let

$$\bar{A}_n = \sum_{m=0}^{2} b_m g_{mn}.$$

From (A1) and (A6) we obtain

$$\bar{A}_{-1} = \frac{2b_0}{a_1} - \left(\frac{2}{a_1}\right)^2 a_0 b_1 + \left(\frac{2}{a_1}\right)^3 a_0 (a_0+1) b_2 - \frac{2b_2}{a_1},$$

$$\bar{A}_0 = \frac{2}{a_1} b_1 - \left(\frac{2}{a_1}\right)(a_0+1) b_2, \quad \bar{A}_1 = \frac{2}{a_1} b_2,$$

$$\bar{A}_3 = \bar{A}_4 = \ldots \bar{A}_{n+3} = 0, \qquad \omega_1 = (\bar{A}_{-1} - \bar{A}_1)/2I'_{a_0}(a_1). \qquad \text{(A7)}$$

The representation of the Green's function in the form (A5) is convenient in the case in which the function $f_1(\theta)$ is a trigonometric polynomial (see (66)). In this case the series (69) converges for any finite values ρ. If $f_1(\theta)$ is the complete Fourier series, it is necessary to use the form (A4). In that case, series (48) converges inside the unit circle of plane ξ (see (38)).

REFERENCES

1. L. V. Kantorovich, Approximate Methods of Higher Analysis [in Russian], FM, 1962.

2. N. N. Lebedev, I. P. Skal'skaya, and Ya. S. Uflyand, Collection of Problems of Mathematical Physics [in Russian], GITTL, 1955.

3. N. N. Lebedev, Special Functions and Their Applications [in Russian], FM, 1963.

4. N. I. Muskhelishvili, Some Basic Problems of the Mathematical Theory of Elasticity [in Russian], Izd. AN SSSR, 1954.

5. N. Ya. Sonin, Investigations of Cylindrical Functions and Special Polynomials [in Russian], GITTL, 1954.

6. A. A. Savelov, Plane Curves [in Russian], FM, 1960.

7. Miln-Thomson, The Calculus of Finite Differences. London, Macmillan, 1951.

8. A. Kratzer, Transcendental Functions [Russian translation], IL, 1963.

9. H. Carslaw, and J. Jaeger, Conduction of Heat in solids [Russian translation], Izd. Nauka, 1964.

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